International Journal of Theoretical Physics, Vol. 43, No. 9, September 2004 (© 2004)

Space-Time Topology (II)—Causality, the Fourth Stiefel–Whitney Class and Space-Time as a Boundary

Malene Steen Nielsen Flagga[1,3] **and Frank Antonsen**[2]

We show that stable causality is related to the vanishing of the top Stiefel–Whitney class of a space-time manifold M, and that if M is a stably causal space-time manifold, then it is the boundary of a five-dimensional space-time. We then propose a scheme for making it both a necessary and sufficient condition.

KEY WORDS: fourth Stiefel–Whitney class; causality; space-time as boundary.

1. INTRODUCTION

Wormholes experienced a renewed interest after 1988. As a consequence of Einstein's General Relativity they pose the possibility of future space-travel and an understanding of quantum gravity. But despite the progress they seem flawed by their apparent ability to become time machines (Visser, 1996).

However, as professor Stephen Hawking once said, there seems to be at least empirical evidence suggesting that the Universe obeys the chronology protection conjecture. There are no time travellers. Antonsen and Bormann (Antonsen and Bormann, 1995a,b; Bormann and Antonsen, 1995) have shown that quantum effects will cause a wormhole attempting to become a time machine to collapse.

This, of course, does not prove that space-time possesses such a global ordering of events. But how does one prove Hawking's chronology protection conjecture? There seems to be no clear agreement among physicists as to what exactly has to be proven, in order to prove that the chronology protection conjecture holds.

Perhaps one needs to start with assumptions on the causality features of space-time, and study the consequences. This is exactly what we have done. Our basic

[1] Bremensgade 33, DK-2300 Copenhagen S, Denmark.
[2] Theoretical Physics Department, University of Groningen, Nijenborgh 4, NL-9747 AG Gronigen, The Netherlands.
[3] To whom correspondence should be addressed at Bremensgade 33, DK-2300 Copenhagen S, Denmark; e-mail: mflagga@villains.de.

assumption is that space-time is *stably causal*, a feature implying the chronology protection conjecture, and this leads to quite interesting results.

If M is indeed stably causal, it is the boundary of a five-dimensional manifold, named V in subsequent sections. This is interesting in connection with Kaluza–Klein theory, where electromagnetism and gravity are unified in exactly five dimensions.

In Antonsen and Flagga (2002) we discussed the first two Stiefel–Whitney classes w_1 and w_2 and showed how w_3 was related to chirality. The starting point was the simple observation, that space-time would only be orientable if w_1 was trivial, and only allow for fermions to exist if w_2 was trivial. Since chiral fermions exist if and only if space-time is orientable and fermions exist, and since the vanishing of w_1 and w_2 leads to the vanishing of w_3 it was obvious there was a relationship.

However w_4 is not trivial merely because w_1, w_2 and w_3 are trivial. The Steenrod Square operation and Wu's formula, which makes $w_3 = 0$ when w_1 and w_2 are trivial, cannot be used for w_4. But we can show, as we hypothesized in our previous article, that it is related to causality, namely to stable causality.

First we review the definition of a physically reasonable space-time manifold, mention a few new properties of this, and introduce the so-called direction-field. Then we show, that if M is stably causal, the fourth Stiefel–Whitney class and the Euler-class $e(M)$ are both trivial.

Since our space-time is non-compact we then proceed to introduce cohomology with compact support, and show that the vanishing of the fourth Stiefel–Whitney class leads M to be the boundary of V. Then we go on to introduce relative cohomology, and show that if M is a boundary, then all Stiefel–Whitney numbers of M are trivial. We then discuss the possibilities of showing, that M as a boundary leads M to be stably causal.

2. A PHYSICALLY REASONABLE SPACETIME

In Antonsen and Flagga (2002), we already discussed the requirements for a *physically reasonable space-time*, M. It is a spacetime that does not disagree with experiments or any deeply held physical principles, such as the Standard Model or General Relativity. While they may not be the full theories, they work within their scopes, and so to model a spacetime, that is in accordance with observation, this places certain restrictions on M.

We defined our spacetime M as a four-dimensional archwise connected, smooth manifold equipped with a Lorentz signature metric $(1, -1, -1, -1)$. The only difference from our previous article is that we shall use $\mathbb{Z}_2$ as the additive group $(\mathbb{Z}_2, +)$.

That M has a pseudo-Riemannian (Lorentzian) metric does not only require M to be paracompact, we also have (Visser, 1996):

Theorem 1. *A manifold M admits a Lorentzian metric if and only if it is (a) paracompact and (b) it admits an everywhere non-vanishing continuous direction field* **or** *M's double cover $\mathcal{D}(M)$ admits an everywhere non-vanishing continuous vector-field.*

A direction field $d(m)$ on a manifold M assigns to each point $m \in M$ a pair of equal but opposite vectors in the tangent space $T_m M$. That is, $m \rightarrow d(m) = \{V_m, -V_m\}$. It is therefore a nowhere-zero vector-field, that is continuous up to a change of sign, and since M is time orientable, the field is time-like.

We can then ask, what exactly is this direction-field? It defines, continuously, a division of nonspace-like vectors into two classes, which we label future- and past-directed, and thereby defines a temporal orientation, making the manifold time-orientable. The lack of continuity simply expresses, that we do not know if M is globally causal.

Locally, however $d(m)$ is continuous. Given $m \in U \subseteq M$, define $I^+(m, U)$ to be the set of points $m' \in M$ which can be reached from m by a curve such that the tangent field $V^\mu \in TU$ is everywhere future directed and time-like (and hence also non-zero).

So locally we can distinguish between future and past, but globally we can only define a division of nonspace-like vectors into the two classes, future- and past-directed. This is the definition of time-orientability giving rise to the existence of the time-like direction-field $d(m) = \{+V|_m, -V|_m\}$. A spacetime-manifold is always locally causal, and globally so, if the direction-field is continuous.

Since our first article we have found, that we can strengthen certain properties of M^4, both topologically and homotopically. Since M is archwise connected we do not need to specify basepoints for the homotopy groups. M^4 is also not only paracompact, it is actually *strongly paracompact* and *finally compact*. The meaning of these, perhaps somewhat unfamiliar, terms are as follows:

Definition 1. A space M is strongly paracompact if each open cover $\mathcal{U} = \{U_i | i \in I\}$ of M has a refinement $\mathcal{U}'$ so that for each $m \in \mathcal{U}'$ there exists an element $U \in \mathcal{U}$ so that:

$$\bigcup \{U' \in \mathcal{U}' | m \in U'\} \subset U.$$

The set on the left hand side is the so-called star of m with respect to $\mathcal{U}'$. Strong paracompactness then states, that this is always contained in $U \in \mathcal{U}$ (Arhangel'skii, 1995). Strongly paracompact thus means, that the sets in the cover themselves only intersect each other a finite number of times. M^4 has this stronger paracompactness feature since it is *separable*. That is M contains an everywhere countable dense subset. This is the case because M^4 is paracompact and contains only one connected component (Bredon, 1993).

An alternative proof is to note that $\mathbb{R}$ is separable since it contains the rational numbers as an everywhere countable, dense subset. Thus also finite powers of $\mathbb{R}$, $\mathbb{R}^n$ leads to separable spaces by the same argument. Any manifold is locally diffeomorphic to $\mathbb{R}^n$. Cover M by an atlas, then each chart U in this atlas is separable when viewed as topological spaces in their own right. If we then denote the everywhere dense subset of U by $\mathbb{D}_U$ it is a countable set, since it is a subset of $\mathbb{Q}^n$. Now let $\mathbb{D} = \cup_U \mathbb{D}_U$, which is then everywhere dense in M itself. It is countable since we have a star-finite atlas.

But a strongly paracompact, regular space is also *finally compact* (Arhangel' skii, 1995):

Definition 2. A space M is finally compact, if each open cover of M has a countable subcover.

So the refinement, $\mathcal{U}'$, consists only of a countable number of sets.

But M^4 is not only regular (or $\mathbf{T}_3$), it is *normal* (or $\mathbf{T}_4$), meaning that for every finite subcover, $\mathcal{U}'$, of M^4 there exists a cover $C = \{V_1, \ldots, V_k\}$ of M^4, so that $\overline{V}_i \subset U_i$, $i = 1, \ldots k$. C is called a *shrinking* of $\mathcal{U}'$.

Definition 3. A space M is said to be *semilocally 1-connected* or *locally relatively simply connected* if each point $m \in M$ has a neighborhood U such that all loops in U are homotopically trivial in M.

An archwise connected space with this property has a simply connected covering space (Bredon, 1993). A space M is semi-locally 1-connected since it is a manifold (Greenberg and Harper, 1981). Intuitively this seems reasonable that a manifold posses this property, since the only exception to homotopic triviality would be those $m \in M^4$ that are singularities, such as the singularity of a black hole. The positions of the singularities are however localisable, once the black hole is located. Their measure on M^4 will be zero, since the number of black holes in M^4 is assumed finite, but of course not constant.

3. CAUSALITY AND THE FOURTH STIEFEL–WHITNEY CLASS

There exists a so-called *primary obstruction class* to a certain cross section over M^4. In general, if M^n is an n-dimensional manifold and if $r = 2l < n$ then the primary obstruction class O_{2l} for the bundle E to posses an every-where non-zero cross-section is equal to the $2l$'th Stiefel–Whitney class (Milnor and Stasheff, 1974):

$$O_{2l} = w_{2l}, 2l < n.$$

If l is odd then the obstruction classes are completely determined by the Stiefel–Whitney classes of the bundle E. When n is even, as in our case, then the highest obstruction class can be identified with the Euler-class $e(M) \in H^n(M; \mathbb{Z})$, provided that M is orientable. Or rather if E is an oriented n-plane bundle over a CW-complex, then $O_n(E)$ is equal to the Euler-class $e(E)$. But since our M^4 is paracompact it possesses a smooth triangulation and can be given the structure of a CW-complex (Milnor and Stasheff, 1974), which is just a certain kind of homological structure.[4]

We will now investigate the fourth Stiefel–Whitney class $w_4(M) \in \check{H}^4(M, \mathbb{Z}_2)$. Like the first, second and third it is the obstruction to a certain section in a certain bundle, and its origin is a Čech-cocycle, i.e. a function $f_4(i_0, i_1, i_2, i_3, i_4) \in \mathbb{Z}_2$ defined on $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_4} \neq \emptyset$, such that $f_4 \in Z^4(M; \mathbb{Z}_2) = \{f \in C^4(M; \mathbb{Z}_2) | \delta f_4 = 0\}$, and therefore defines an element $w_4 = [f_4] \in H^4(M; \mathbb{Z}_2)$.

A k-plane bundle is a k-dimensional subspace of a bundle. For instance, if we view the frame-bundle FM not as principal-bundle but as the space of vier-beins, then the subspaces would be drei-, zwei- and ein-beins. And these span the k-plane bundles of FM.

Theorem 2. *For a k-plane bundle:*

1. $w_1 = 0 \Leftrightarrow$ *the vector bundle is orientable, and*
2. w_k *is the mod 2 reduction of the Euler class, e.*

For a proof see, for example (Bredon, 1993). So we see that, in our case, $w_4(M) = e(M)$ mod 2. Both classes have the property, that if M possesses a globally defined nowhere-zero section, then they are trivial (Milnor and Stasheff, 1974). The converse is only true for $e(M)$ under certain circumstances, which we will investigate in subsequent sections. So, as we already mentioned, the top Stiefel–Whitney class is not the real obstruction when $k = n$ the dimension of M.

These were purely topological considerations, we now proceed to study the possible causality features of M^4. One of these is the so-called *strong causality condition* (SCC). If M has this feature, every $m \in M$ has a neighborhood containing a neighborhood of m not intersected by nonspace-like curves more than once (a *local causality neighborhood)*. If the strong causality condition holds, one can determine the topological structure of space-time by observing causal relationships (Hawking and Ellis, 1973).

The SCC is enough to exclude closed time-like curves on M, but not strong enough to rule out all causal pathologies. As Hawking and Ellis point out, even with this condition we can still have a space-time which is on the verge of violating the chronology protection conjecture, because the slightest variation of the metric can lead to closed time-like curves. This condition is so to speak not stable.

To be physically significant a property of space-time must have some form of stability. That is, it should also be a property of nearby space-times. General

[4] We will return to the details of the CW-complexes in subsequent sections.

Relativity should be the limit of a theory of quantum gravity and in such a theory the Uncertainty Principle would prevent the metric from having exact values at every point. So if it only takes a slight variation of the metric of a strongly causal space-time to cause closed time-like curves to exist, the Uncertainty Principle says, they may indeed be there.

If one defines a topology on the sets of all spacetimes, a meaning to the word "nearby" for the metric can be given. The so-called SCC holds on M, if the space-time metric g has an open neighborhood in the C^0-open topology on $\mathcal{T}^0_{2(S)}(M)$, such that there are no closed time-like curves in any metric, belonging to that neighborhood. In other words, it is possible to slightly expand the light-cones at every point without introducing closed time-like curves.

A stably causal space-time is also strongly causal. And M is stably causal if, and only if, there is a function on M whose gradient is everywhere time-like (Hawking and Ellis, 1974). In other words, the time-like direction field is now continuous.

Theorem 3. *Let M^4 be given as above. If M^4 is stably causal then $e(M) = 0$, $w_4 = 0$ and $\chi(M) = 0$.*

Proof: If M is stably causal then there exists a global $\mathbb{R}$-valued function $f_H \in \Omega^0(M) = \mathcal{F}(M)$, with the property that its gradient is everywhere time-like. The gradient of f_H can be read off from $df_H \in \Omega^1(M)$ and is the vector $V_H = (\partial_0 f_H, \partial_1 f_H, \partial_2 f_H, \partial_3 f_H) \in \mathcal{X}(M)$.

In other words $(df_H)_\mu = g_{\mu\nu}V_H^\nu$. That V_H is time-like means that for all $m \in M$:

$$g_m(V_H, V_H) = g_{\mu\nu}(m)V_H^\mu V_H^\nu > 0$$

In particular, V_H is everywhere non-zero. This means that M has a non-zero section in *TM* and then $e(M) = 0$ as well as $w_4 = 0$.

The dual one-form of V_H is $\omega_\nu g^{\mu\nu}V_H^\mu$. Since V_H is a global gradient field by definition $d\omega_n u = 0$, and so $*d\omega_\nu = 0$, it is curl-free, so $\chi(M) = 0$ (Bass and Witten, 1957). □

As soon as we introduce cohomology with compact support, $\chi(M) = 0$ because $e(M) = 0$ by Stokes theorem.

3.1. A Note on the Euler-Class

We wish to study the classes for the tangent bundle *TM* of M and will use the standard notation $e(TM) = e(M)$. There are several points worth noting now.

In the literature there are many references to $e(M)$, but some confusion as to the requirements for its existence. In (Husemoller, 1994) the Euler-class is defined as:

$$e = (\pi^*)^{-1} j^*(u) \in H^n(M; \mathbb{Z})$$

where n is the dimension of the bundle E, $\pi : E \to M$, and u is the so-called *fundamental class*. For E the set E_0 denotes the open subset of non-zero vectors, and i is the inclusion $j : E \to (E, E_0)$, so that $j^* : H^r(E, E_0) \to H^r(E)$. Only when the bundle is oriented and the coefficients are integers does this class carry the name Euler-class.

In physics, however, such as Nakahara (1990), the requirements are typically for both the bundle to be oriented and of *even* dimension. When M is even-dimensional the transition functions t_{ij} are even dimensional matrices and therefore their determinants possess an $SO(n)$-invariant square-root known as the *Pfaffian* (which requires the reduction of the structure-group from $O(n)$ to $SO(n)$ to be well-defined), $\det(t_{ij}) = \mathrm{Pf}(t_{ij})^2$. Since in our case w_4 is the modulo 2 reduction, we need only define $e(M)$ up to a sign, and use the fact that the square of $e(M)$ is the top Pontrjagin class $p_2(M)$ of TM. That is:

$$e(M) \wedge e(M) = P_2(M) \in H^8(M; \mathbb{R}).$$

Note the abuse of notation: Both sides of the equation should be understood as functions of a 4×4 matrix A and not of the curvature two-form R, since $p_2(R)$ vanishes (Nakahara, 1990). The skew-symmetric matrix, whose determinant we wish to calculate to get the Pfaffian, is of course related to the Riemann-curvature tensor, but we're interested in it here as a two-form $R = \frac{1}{2} R_{ab} \theta^a \wedge \theta^b$, where $\{\theta^a\}$ is the no-coordinate basis $\theta^a = e^a_\mu dx^\mu$.

This gives us that

$$p_2(M) = \left(\frac{1}{2\pi}\right)^4 \det R$$

In terms of the curvature two-form R we have:

$$e(M) = Pf(R) = \frac{1}{32\pi^2} \varepsilon^{abcd} R_{ab} \wedge R_{cd}$$

Note that here the coefficients are real and not integers. That $e(M)$ can be written as the square of another class, translate mathematically to the following: If we again let u denote the fundamental class and $\phi : H^r(M) \to H^{r+n}(E, E_0)$, the expression $e(M) = \phi^{-1}(u^2)$ holds for the Euler-class (Husemoller, 1994).

4. PARACOBORDISM

We have now seen, how equipping M^4 with the stable causality feature leads to the triviality of the Euler-class and the fourth Stiefel–Whitney class. But this has further consequences if we use cohomology with compact support. In that case, M^4 turns out to be the boundary of a five-dimensional manifold, to be called V.

4.1. Cohomology with Compact Support

Since M^4 is not compact we cannot *a priori* define the Poincare duality, nor the integration of characteristic classes. However, we can utilize several properties of M^4 to define, in a coherent manner, cohomology with compact support.

First note that, as M is normal and paracompact, each locally finite cover has a shrinking. Then any open non-empty subset of $\mathbb{R}^4$, and by diffeomorphism also M^4, will have a sequence of compact sets $\{K_j\}_{j\in\mathbb{N}}$ (Choquet-Bruhat *et al.*, 1982) so that:

$$k_1 \subset K_2^\circ \subset K_2 \subset \cdots \subset K_j^\circ \subset K_j \ldots ; \bigcup_{J\in\mathbb{N}} K_j^\circ = M$$

This sequence also exists for M^4 if we let $\{U_i\}$ be a cover of open, bounded sets—the sets $K_i = \bar{U}_i$ then fulfills the above requirements.

Since the interiors of the K_j's form an open cover of M^4, any compact $K \subset M^4$ will be contained in some k_j from a certain j. Then the $C^\infty_{K_j}(M)$ (functions on M^4 with support in k_j) are Frechét spaces.[5]

The space $C^\infty_c(M)$ of functions on M with compact support in M will play a central role in the following. We can equip this space with the inductive limit topology by viewing it as:

$$C^\infty_c(M) = \bigcup_{j=1}^{\infty} C^\infty_{K_j}(M)$$

A sequence in $C^\infty_c(M)$ is however only a Cauchy sequence if the elements of the sequence all have compact support in a certain set K_j. But if it does, then it will converge. This space is also known as $\Omega^0_c(M)$, the space of 0-forms with compact support on M.

On each of the compact subsets k_j of M^4 we can restrict our $\mathbb{Z}$-orientation to k_j. We can then use the isomorphism:

$$H_4(M, M - k_j; \mathbb{Z}) \to \Gamma K, \quad \text{the set of all sections over} \quad K_j \tag{1}$$

to get the *fundamental homology class* $o_K \in H_4(M, M - K_j; \mathbb{Z})$. With this singular cohomology with compact support is defined as (Greenberg and Harper, 1981):

$$H^r_c(M; \mathbb{Z}) = \lim_{K_j \text{ compact}} H^r(M, M - K_j; \mathbb{Z}) \tag{2}$$

[5] A Frechét space is locally convex, metrizable, have a translation invariant metric and all Cauchy sequences converge.

A cohomology class $\omega(m)$ in $H_c^r(M;\mathbb{Z})$ is represented by a co-chain vanishing off some compact subset K, i.e. the cochain annihilates all chains with support in $M - K$. This is a natural generalization to a non-compact space, as it ensures sufficiently rapid fall-off at infinity. With this definition integrals are well defined.

This can also be seen since $C_c^\infty(M)$ is dense in both $L_p(M)$, $p \in [1;\infty]$ and the space $L_{p,\text{loc}} = \{f \text{ measurable}|f|_K \in L_p(K) \text{ for } K \text{ compact} \subset M\}$ (Choquet-Bruhat *et al.*, 1982).

However, only *proper maps* induce homomorphisms in cohomology with compact support. If $f : M \to N$, we must require that for $L \subset N$ and compact $f^{-1}(L)$ is a compact subset of M. The pullback of the inclusion map i, for instance, is proper.

Consider now a topological space M, and let $\Delta : M \to M \times M$ be the diagonal map, $m \mapsto (m, m)$. This induces a map in cohomology $\Delta^* : H^r(M \times M) \to H^r(M)$. Together with the Künneth map $H^r(M) \times H^r(M) \to H^{r+q}(M \times M)$, this gives rise to a product on $H^*(M)$ by composition $H^r(M) \times H^q(M) \to H^{r+q}(M \times M) \to H^{r+q}(M)$, denoted by $(a, b) \mapsto a \cup b$, $a \in H^r$, $b \in H^q$ and called the *cup product*. With this product, $H^*(M)$ becomes a ring for every topological space X.

If we again let K denote a compact subset and U an open subset of M^4, with $K \subset U$, there is a diagram involving the cup-product:

$$\begin{array}{lcl} H^r(M) \otimes H^{4-r}(M, M-K) & \overset{\text{cup}}{\to} & H^4(M, M-K) \\ j^* \otimes 1 \downarrow & & \downarrow j^* \\ H^r(U) \otimes H^{4-r}(M, M-K) & \overset{\text{cup}}{\to} & H^4(U, U-K) \end{array}$$

Note that the inclusion map $i : (U, U - K) \to (M, M - K)$ is an excision which induces an isomorphism in both singular homology and cohomology theory.

Now if $\langle , \rangle$: $H^*(V; M) \otimes H_*(V; M) \to \mathbb{Z}$ or $\mathbb{Z}_2$ is the canonical pairing induced by the substitution pairing $C^*(V, M) \otimes C_*(V, M) \to \mathbb{Z}$ or $\mathbb{Z}_2$ then there is a relation between the cup-product between two cochains and the *cap product* between a chain and co-chain.

The cap-product is defined on as:

$$\cap : H^r(M) \otimes H_q(M) \to H_{r-q}(M)$$

where

$$u \cap c = (1 \otimes u)\delta c$$

with δ some diagonal approximation. If $c \in H_4(U, U - K)$ and $v \in H^1(U)$ while $v \in H^{4-k}(U)$ then:

$$\langle u \cup v, c\rangle = \langle u, v \cap c\rangle$$

This holds for cohomology in general, so also for compactly supported cohomology.

The cap-product has the property that if $f \in C^r(M)$ then (Husemoller, 1994):

$$\partial(f \cup c) = \delta f \cup c + (-1)^r f \cup \partial c \tag{3}$$

Using compact supported homology the homomorphism:

$$D : H_c^r(M) \to H_{4-r}(M)$$

is an isomorphism and we thusly have Poincaré duality (Greenberg and Harper, 1981). The cup product also induces, by passage to the limit, a cup-product:

$$H_c^r(M) \times H_c^q(M) \to H_c^{r+q}(M)$$

so the direct sum $\otimes H_c^r(M) = H_c^*(M)$ becomes an R-algebra, where R is the orientation, which does not, in our case possess a natural identity element. But it is contravariantly functorial with respect to proper maps, that is, if $f : A \to B$ gets mapped to $H^*(A) \to H^*(B)$ and proper maps induces homomorphisms.

We must therefore define an extension $\tilde{H}_c^*$ to get a unital algebra, which is one of the subjects of our current research.

Since M^4 is connected the generator of $H_c^4(M)$ corresponds to the canonical generator $H_0(m)$ under the duality isomorphism and is the canonical class of the $\mathbb{Z}$-orientation on M^4, so $H_c^4(M) = \mathbb{Z}$.

However, while we have Poincaré duality we do not *a priori* have finite homology groups. Only for $K \subset M$ compact are the $H_r(K)$'s finitely generated. Obviously we also have Poincaré duality for the compact subsets of M^4. What we do have is that since M^4 is paracompact it can be embedded in $\mathbb{R}^8$ (Bredon, 1993) and is thusly an *Euclidean Neighborhood Retract*, meaning:

$$\check{H}^*(M) = \lim H^*(K_j) \to H^*(M)$$

where K_j ranges over the neighborhoods of M^4 in $\mathbb{R}^8$.

4.2. The Stiefel–Whitney Numbers

The *Stiefel–Whitney numbers* of our M^4 are formed using a monomial $\varpi = w_1^{r(1)} \dots w_4^{r(4)}$ of degree 4 and evaluating it on the orientation class o_M of M:

$$\langle \varpi, o_M \rangle \tag{4}$$

An n-dimensional manifold has one *Stiefel–Whitney number* for each sequence $r(1), \dots, r(n)$ where $n = r(1) + 2r(2) + \dots + nr(n)$. In our case we would get the sequences (4, 0, 0, 0), (0, 2, 0, 0), (1, 0, 1, 0) and (0, 0, 0, 1) (Milnor and Stasheff, 1974).

Now, let M be stably causal and take Čech-cohomology with compact support $\check{H}^r_c(M, \mathbb{Z}_2)$. This means that the Čech-cochains are defined on the finite intersection of compact sets K_j, where $K_j^\circ = U_j$ open sets.

Since $w_1 = w_2 = w_3 = w_4 = 0$ all the Stiefel–Whitney numbers of our M^4 are zero. That is

$$\langle \varphi, o_M \rangle = 0, \tag{5}$$

which means that either ϖ is exact or o_M is a boundary. Now, φ is not exact, so $\exists V : o_M = \partial o_V$. Since M is paracompact and closed in the topology of V, V must itself be paracompact and oriented. We have shown:

Theorem 4. *If M^4 is stably causal then M^4 is the boundary of some paracompact and oriented 5-manifold, V*

5. BOUNDARIES AND CAUSALITY

We have shown that a sufficient condition for M^4 to be the boundary of V, is that M^4 is stably causal and equipped with cohomology with compact support. But is this a necessary condition? Can we show, that simply by assuming that M is a boundary of a five-dimensional manifold, then it must necessarily be stably causal? *A priori* we know nothing of this five-dimensional manifold V, save that its boundary is M^4. What is its topological and geometrical properties, and what is this fifth dimension, which we cannot see nor interact with?

Assume V is paracompact, archwise connected and oriented, that $M^4 = \partial V$ and that both spaces have cohomology with compact support. We can then study what is known as **relative cohomology** of $M^4 \subset V$. In the following section we will review a few fundamental results and discuss them in our case.

5.1. Relative Cohomology

In the following $i : M \hookrightarrow V$, denotes the inclusion, $i^* : H^r(V) \to H^r(M)$, and $i^*_M : H_5(V, V - M) \to \Gamma_c M = C^\infty_c(M)$. However, since M is a connected and non-compact subset of V, $H_5(V, V - M) = 0$ (Greenberg and Harper, 1981). Note that also $i^*_V : H_5(V) \xrightarrow{\sim} C^\infty_c(V)$ and that $H_r(V) = 0$ for $r > 5$. We also have that $H_r(V, V - M) = 0$ for $r > 5$ and $H_5(V) = 0$. For the pair (V, M) there is also the so-called *connecting morphism* (Greenberg and Harper, 1981):

$$\delta : H^r(M) \to H^{r+1}(V, M)$$

Consider now the diagram on the co-chain level:

$$\begin{array}{ccccccccc}
0 \to & C^r(V, M) & \to & C^r(V) & \xrightarrow{i^*} & C^r(M) & \to 0 \\
& \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\
0 \to & C^{r+1}(V, M) & \to & C^{r+1}(V) & \to & C^{r+1}(M) & \to 0
\end{array}$$

If u is a cochain on V, $u \in H^r(V)$, such that $i^*(u)$ is a cocycle on M, which represents a cohomology class $\bar{u} \in H^r(M)$. Since $i^*(u)$ is a cocycle, $\delta(i^*u) = i^*(\delta u) = 0$, δu is a relative $(r+1)$-cocycle on the pair (V, M) and therefore represents a cohomology class $\overline{(\delta u)} \in H^{r+1}(M, V)$. This class is independent of the choice of u, so $\delta\bar{u} = \overline{\delta u}$.

Thus, we have the exact cohomology sequence (Greenberg and Harper, 1981):

$$0 \to H^0(V, M) \to \cdots \to H^r(V) \xrightarrow{i^*} H^r(M) \xrightarrow{\delta} H^r(V, M) \to \cdots.$$

When M is the boundary of V we have that M is oriented. Because on V, a manifold with M as boundary, Stoke's Theorem is valid (Bredon, 1993). If we have a boundary point v we can take a system of local coordinates $\{x_0, \ldots, x_4\}$ such that V is given by $x_4 \leq 0$ and $\{x_0, \ldots x_3\}$ form local coordinates for the boundary. We can takes these coordinates to define an orientation on the boundary, $\partial V = M$.

This means that if $o_V \in H_5(V, M)$ is the $\mathbb{Z}_2$ orientation on V relative to $M(V$ is orientable along $M)$ a homeomorphism $\phi : \pi^{-1}(M) \to M \times \mathbb{Z}$ exists, and the diagram:

$$\begin{tikzcd} \pi^{-1}(M) \arrow[rr, "\phi"] \arrow[dr] & & M \times Z \arrow[dl] \\ & M & \end{tikzcd} \tag{6}$$

commutes.

If ω denotes a four-form on V with compact support, then $i^*\omega$ is a four-form on M, Stoke's theorem states that:

$$\int_V d\omega = \int_M \omega \tag{7}$$

The cap-product, which is the Poincaré duality, also exists for relative cohomology, yielding (Bredon, 1993)

$$\cap : H^r(V) \otimes H_5(V, M) \to H_{5-r}(V.M)$$

and

$$\cap : H^r(V, M) \otimes H_4(V, M) \to H_{5-r}(V)$$

And generally, since M is the boundary of V:

$$H_r(V, M) \simeq H^{5-r}(V) \text{ and } H_r(V) \simeq H^{5-r}(V, M)$$

Letting o_V in $H_5(V.M)$ denote the $\mathbb{Z}_2$-orientation on V, $\partial o_V = o_M \in H_4(M)$ will be the $\mathbb{Z}_2$-orientation of M. Equation (7) can then be written in compact form for the 4-form ω on V:

$$\langle d\omega, o_V \rangle_V = \langle \omega, o_M \rangle_M \tag{8}$$

The *period* of a closed r-form ω over a cycle c is generally defined as $\langle \omega, c \rangle$. This period vanishes if ω is exact or if c is a boundary. So obviously, if $M = \partial V$, then $\langle \omega, o_M \rangle = 0$ since o_M is a boundary.

Theorem 5. *Let M be as above, so that $M = \partial V$. Then all the Stiefel–Whitney numbers of M are zero.*

Proof: Let o_V be the $\mathbb{Z}_2$-orientation class of V, where we have that $o_V \in H_5(V, M)$. Then we have that $\partial o_V = o_M \in H_4(M)$ is the orientation class on M. For the monomial $\varpi = w_1^{r(1)} w_2^{r(2)} w_3^{r(3)} w_4^{r(4)}$, Stoke tells us that:

$$\langle \delta \varpi, o_V \rangle = \langle \varpi, o_M \rangle \tag{9}$$

We have that $TV_M = TM \oplus L^1$, so $i^*(w_r(TV)) = w_r(TM) = w_r(M)$. In the exact sequence:

$$H^4(V) \xrightarrow{i^*} H^4(M) \xrightarrow{\delta} H^5(V, M)$$

we have that $\delta i^* = 0$, as we saw above, so $\delta \varpi = 0$. This means that $\langle \delta \varpi, o_V \rangle = 0$ for all choices of $4 = r(1) + 2r(2) + 3r(3) + 4r(4)$ and all the Stiefel–Whitney numbers of M are zero. □

6. CONCLUSION

We have seen, that all four Stiefel–Whitney classes, which are elements of the Čech-cohomology, have physical significance. The relationship between global causality, the fourth Stiefel–Whitney class and M as a boundary of a five-dimensional manifold concludes this. This is interesting in the current discussion on how many dimensions our universe really has.

We were able to show that a stably causal manifold has $w_4 = 0$. We also showed that such a manifold was the boundary of a five-manifold. This, however is not enough to ensure that w_4 is itself trivial. All we know, is that $\langle w_4, o_M \rangle = 0$. What we need is for this to give us a no-where zero time-like vector field, which is the gradient of a globally defined function. To get such a function, we need the Euler class to vanish, $e(M) = 0$, since this would lead to the existence of a nowhere-zero section over all of M. This section must then be shown to be time-like and finally a global gradient field. Since $w_2 = 0$, M is parallelizable and there is a diffeomorphism $f : TM \to M \times \mathbb{R}^4$ such that each $T_m M$ is carried linearly isomorphically onto $\{m\} \times \mathbb{R}^4$. *TM* is therefore isomorphic to the trivial 4-plane bundle leading to the following theorem (Bredon, 1993):

Theorem 6. *If E is en orientable n-plane bundle over the n-dimensional complex M then E has a nonzero-section if and only if $0 = e \in H^n(M)$.*

We are currently working on using this theorem, together with imposing a causal structure on the cohomology modules induced by the causal structure on M, to try to show that a vanishing top Stiefel–Whitney class ensures stable causality.

In the early 1920s, Kaluza and Klein introduced a fifth dimension, which is curved by electromagnetic potentials. Their ideas were based on the following analogy: in general relativity, distances depend locally on the gravitational potential; one might therefore imagine new dimensions such that the generalized distance depends also on the electromagnetic potential. This may lead to a unified theory of gravity and electromagnetism. This idea has been revived later in the context of string-theory, where, however, the extra spatial dimensions are believed to be compact and microscopic. The fifth dimension in this paper, related to stable causality, is macroscopic however, and therefore more in line with Kaluza and Kleins original ideas, where the fifth dimension, although small, was not necessarily of Planck scale size. But, in a sense, the five-manifold appearing here is completely different, "orthogonal" even, to the Kaluza-Klein or superstring theory extra dimensions—it is not compactified, and physical four-dimensional spacetime arises as the boundary. This opens up a lot of questions as to the physical effects of such a hypothetical extra dimension.

REFERENCES

Antonsen, F. and Bormann, C. (1995a). The Casimir effect and the stability of space-time, In *Proceedings of The Third Alexander Friedmann International Seminar on Gravitation and Cosmology*, Friedmann Laboratory Publishing.

Antonsen, F. and Bormann, C. (1995b). Wormholes and time-machines, In *Proceedings of The Third Alexander Friedmann International Seminar on Gravitation and Cosmology*, Friedmann Laboratory Publishing, St. Petersburg.

Antonsen, F. and Flagga, M. S. N. (2002). Space-time topology (I) – Chirality and the third Stiefel-Whitney class, *International Journal of Theoretical Physics* **41**(2), 171–198.

Arhanhel'skii, A. V. (1995). *General Topology III*, Springer, Berlin.

Bass, R. W. and Witten, L. (1957). *Review on Modern Physics* **29**(3), 452–453.

Bormann, C. and Antonsen, C. (1995). The Casimir effect of curved space-time, *Proceedings of The Third Alexander Friedmann International Seminar on Gravitation and Cosmology*, Friedmann Laboratory Publishing, St. Petersburg.

Bredon, G. E. (1993). *Topology and Geometry*, Springer, Berlin.

Choquet-Bruhat, Y., DeWitt-Morette, C., and Dillard-Bleick, M. (1982). *Analysis, Manifolds and Physics, revised edn.*, North-Holland, New York.

Greenberg, M. J. and Harper, J. R. (1981). *Algebraic Topology – A First Course*, Mathematics Lecture Note Series, Perseus Books Group, Boulder, CO.

Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.

Husemoller, D. (1994). *Fibre Bundles*, Spinger-Verlag, Berlin.

Milnor, J. W. and Stasheff, J. D. (1974). *Characteristic Classes*, Princeton University Press, Princeton.

Nakahara, M. (1990). *Geometry, Topology and Physics*, IOP Publishing, Bristol.

Visser, M. (1996). *Lorentzian Wormholes*, AIP Press, New York.